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Stable and efficient differential estimators on oriented point clouds Supplementary material

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Contents

In this document, we detail the proofs for the main contribution of the article: Theorems 1 (Section 1) and 2 (Section 2). We also provide additional results where we have considered various settings for the radius r and the noise level. We also provide a test on a non-uniformly sampled surface (Section 4).

1. Proof of Theorem 1 - Normalized parameters of the fitted algebraic sphere

This section contains the proof of Theorem 1 that relates the normalized parameters $\hat{u}_c$, $\hat{\mathbf{u}}_\ell$ and $\hat{u}_q$ of the fitted algebraic sphere to the differential properties of the surface. We first give the asymptotic expression of various differential quantities involved in the algebraic sphere regression. Then we integrate these quantities, and we finally assemble the results to obtain Taylor expansions of the sphere parameters.

1.1. Differential quantities

We first give the Taylor polynomials of the coordinates $\mathbf{f}$, the normal vectors $\mathbf{n}$ and their dot product in the local principal frame. Using polar coordinates $(\rho,\theta) \in (0,r)\times(0,2\pi)$, these quantities are given in the form of polynomials of variable ρ with coefficients depending on variable θ. The coefficients also contain the different derivatives $a_{k,j-k}$ of the surface height defined by Equation 10.

Coordinates. The surface of Equation 9 is expressed in polar coordinates by

$$\mathbf{f}(\rho,\theta) = \begin{bmatrix}\rho\cos(\theta) & \rho\sin(\theta) & z(\rho,\theta)\end{bmatrix}^T. \tag{1}$$

The Taylor expansion of the height field function z in Equation 10 is written in polar coordinates as $z(\rho,\theta) = \sum_{k=2}^{5} \rho^k b_k(\theta) + O(\rho^6)$ with the coefficients b_k equal to

$$
\begin{aligned}
b_2(\theta) &= \frac{1}{2}\left(\kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta)\right), \\
b_3(\theta) &= \frac{1}{6}\left(a_{30}\cos^3(\theta) + 3a_{21}\cos^2(\theta)\sin(\theta) + 3a_{12}\cos(\theta)\sin^2(\theta) + a_{03}\sin^3(\theta)\right), \\
b_4(\theta) &= \frac{1}{24}\left(a_{40}\cos^4(\theta) + 4a_{31}\cos^3(\theta)\sin(\theta) + 2a_{22}\cos^2(\theta)\sin^2(\theta) + 4a_{13}\cos(\theta)\sin^3(\theta) + a_{04}\sin^4(\theta)\right), \\
b_5(\theta) &= \frac{1}{120}\left(a_{50}\cos^5(\theta) + 5a_{41}\cos^4(\theta)\sin(\theta) + 10a_{32}\cos^3(\theta)\sin^2(\theta) + 10a_{23}\cos^2(\theta)\sin^3(\theta) + 5a_{14}\cos(\theta)\sin^4(\theta) + a_{05}\sin^5(\theta)\right).
\end{aligned}
$$

The squared height (that is required latter) is $z(\rho,\theta)^2 = \sum_{k=4}^{7} c_k(\theta)\rho^k + O(\rho^8)$ with coefficients $c_4(\theta) = b_2(\theta)^2$, $c_5(\theta) = 2b_2(\theta)b_3(\theta)$, $c_6(\theta) = b_3(\theta)^2 + 2b_2(\theta)b_4(\theta)$, and $c_7(\theta) = 2b_2(\theta)b_5(\theta) + 2b_3(\theta)b_4(\theta)$.

Tangents Before introducing the normal vectors, the Taylor polynomials of the tangents are required. In the principal frame, the partial derivatives of $\mathbf{f}$ with respect to x and y (denoted $\partial_x\mathbf{f}(x,y)$ and $\partial_y\mathbf{f}(x,y)$ respectively) are given by $\partial_x\mathbf{f}(x,y) = \begin{bmatrix}1 & 0 & \partial_x z(x,y)\end{bmatrix}^T$ and $\partial_y\mathbf{f}(x,y) = \begin{bmatrix}0 & 1 & \partial_y z(x,y)\end{bmatrix}^T$. In polar coordinates, the partial derivatives of z with respect to x and y are $\partial_x z(\rho,\theta) = \sum_{k=1}^{4} d_{xk}(\theta)\rho^k + O(\rho^5)$ and $\partial_y z(\rho,\theta) = \sum_{k=1}^{4} d_{y_k}(\theta)\rho^k + O(\rho^5)$ with the following coefficients

$$
\begin{aligned}
d_{x1}(\theta) &= \kappa_1 \cos^2(\theta), \\
d_{x2}(\theta) &= \frac{1}{2}\left(a_{30}\cos^2(\theta) + 2a_{21}\cos(\theta)\sin(\theta) + a_{12}\sin^2(\theta)\right), \\
d_{x3}(\theta) &= \frac{1}{6}\left(a_{40}\cos^3(\theta) + 3a_{31}\cos^2(\theta)\sin(\theta) + 3a_{22}\cos(\theta)\sin^2(\theta) + a_{13}\sin^3(\theta)\right), \\
d_{x4}(\theta) &= \frac{1}{24}\left(a_{50}\cos^4(\theta) + 4a_{41}\cos^3(\theta)\sin(\theta) + 6a_{32}\cos^2(\theta)\sin^2(\theta) + 4a_{23}\cos(\theta)\sin^3(\theta) + a_{14}\sin^4(\theta)\right).
\end{aligned}
$$

$$
\begin{aligned}
d_{y_1}(\theta) &= \kappa_2 \sin^2(\theta), \\
d_{y_2}(\theta) &= \frac{1}{2}\left(a_{21}\cos^2(\theta) + 2a_{12}\cos(\theta)\sin(\theta) + a_{03}\sin^2(\theta)\right), \\
d_{y_3}(\theta) &= \frac{1}{6}\left(a_{31}\cos^3(\theta) + 3a_{22}\cos^2(\theta)\sin(\theta) + 3a_{13}\cos(\theta)\sin^2(\theta) + a_{04}\sin^3(\theta)\right), \\
d_{y_4}(\theta) &= \frac{1}{24}\left(a_{41}\cos^4(\theta) + 4a_{32}\cos^3(\theta)\sin(\theta) + 6a_{23}\cos^2(\theta)\sin^2(\theta) + 4a_{14}\cos(\theta)\sin^3(\theta) + a_{05}\sin^4(\theta)\right).
\end{aligned}
$$

The squared partial derivative of z with respect to x in polar coordinates is $\partial_x z(\rho,\theta)^2 = \sum_{k=2}^{5} e_{xk}(\theta)\rho^k + O(\rho^6)$ with coefficients $e_{x2}(\theta) = d_{x1}(\theta)^2$, $e_{x3}(\theta) = 2d_{x1}(\theta)d_{x2}(\theta)$, $e_{x4}(\theta) = d_{x2}(\theta)^2 + 2d_{x1}(\theta)d_{x3}(\theta)$, and $e_{x5}(\theta) = 2d_{x1}(\theta)d_{x4}(\theta) + 2d_{x2}(\theta)d_{x3}(\theta)$. The formula for $\partial_y z$ and its associated coefficients e_{y_k} are the same as $\partial_x z$ and e_{xk} using y subscript instead of x.

Normal vectors. We denote by $\mathbf{v}$ a vector orthogonal to the surface $\mathbf{v}(x,y) = \partial_x\mathbf{f}(x,y) \times \partial_y\mathbf{f}(x,y)$, which is equal to $\begin{bmatrix}-\partial_x z(x,y) & -\partial_y z(x,y) & 1\end{bmatrix}^T$, so that the normal vector $\mathbf{n}$ is given by $\mathbf{n}(x,y) = \frac{\mathbf{v}(x,y)}{\|\mathbf{v}(x,y)\|}$. The squared norm of $\mathbf{v}$ is $\|\mathbf{v}(\rho,\theta)\|^2 = 1 + \sum_{k=2}^{5} f_k(\theta)\rho^k + O(\rho^6)$, with $f_k(\theta) = e_{xk}(\theta) + e_{y_k}(\theta)$. Using the Taylor expansion of $1/\sqrt{1+X}$, the inverse of the norm is approximated by $1/\|\mathbf{v}(\rho,\theta)\| = 1 + \sum_{k=2}^{5} g_k(\theta)\rho^k + O(\rho^6)$, with $g_2(\theta) = -\frac{1}{2}f_2(\theta)$, $g_3(\theta) = -\frac{1}{2}f_3(\theta)$, $g_4(\theta) = \frac{1}{8}\left(3f_2(\theta)^2 - 4f_4(\theta)\right)$, and $g_5(\theta) = \frac{1}{4}\left(3f_2(\theta)f_3(\theta) - 2f_5(\theta)\right)$.

Finally the normal vector $\mathbf{n}$ is asymptotically equivalent to

$$
\mathbf{n}(\rho,\theta) = \begin{bmatrix}\mathbf{n}_x(\rho,\theta) \\ \mathbf{n}_y(\rho,\theta) \\ \mathbf{n}_z(\rho,\theta)\end{bmatrix} = \begin{bmatrix}\sum_{k=1}^{4} h_{xk}(\theta)\rho^k + O(\rho^5) \\ \sum_{k=1}^{4} h_{y_k}(\theta)\rho^k + O(\rho^5) \\ 1 + \sum_{k=2}^{5} g_k(\theta)\rho^k + O(\rho^6)\end{bmatrix}, \tag{2}
$$

with $h_{x1}(\theta) = -d_{x2}(\theta)$, $h_{x2}(\theta) = -d_{x3}(\theta)$, $h_{x3}(\theta) = -d_{x4}(\theta) - g_2(\theta)d_{x2}(\theta)$, $h_{x4}(\theta) = -d_{x5}(\theta) - g_2(\theta)d_{x3}(\theta) - g_3(\theta)d_{x2}(\theta)$, and using similar formula for $h_{y_k}(\theta)$ using y subscript.

Dot products. The asymptotic dot product between the coordinates and the normal vectors is $\mathbf{f}(\rho,\theta)\,.\,\mathbf{n}(\rho,\theta) = \sum_{k=2}^{5} m_k(\theta)\rho^k + O(\rho^6)$ with the following coefficients

$$
\begin{aligned}
m_2(\theta) &= \cos(\theta)h_{x1}(\theta) + \sin(\theta)h_{y_1}(\theta) + b_2(\theta),\\
m_3(\theta) &= \cos(\theta)h_{x2}(\theta) + \sin(\theta)h_{y_2}(\theta) + b_3(\theta),\\
m_4(\theta) &= \cos(\theta)h_{x3}(\theta) + \sin(\theta)h_{y_3}(\theta) + b_4(\theta) + g_2(\theta)b_2(\theta),\\
m_5(\theta) &= \cos(\theta)h_{x4}(\theta) + \sin(\theta)h_{y_4}(\theta) + b_5(\theta) + g_2(\theta)b_3(\theta) + g_3(\theta)b_2(\theta).
\end{aligned}
$$

The dot product of the coordinates with themselves, which is the squared norm of the positions, is $\|\mathbf{f}(\rho,\theta)\|^2 = \mathbf{f}(\rho,\theta)\,.\,\mathbf{f}(\rho,\theta) = \rho^2 + z(\rho,\theta)^2 = \rho^2 + \sum_{k=4}^{7} c_k(\theta)\rho^k + O(\rho^8)$.

1.2. Integrated quantities

We now give results of the integration of the previous quantities over the cylindrical neighborhood (see Equation 11). These calculations are technical but fairly straightforward since they only involve polynomial integrations. Moreover, many integrals containing coefficients of the form $\cos^p(\theta)\sin^q(\theta)$ are discarded when p or q are odd. On the other hand, the coefficients are often tedious to write so we only give the results.

Coordinates. The integration over $\mathcal{D}_r$ of the coordinates $\mathbf{f}$ of Equation 1 results in $\iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\rho d\rho d\theta = \begin{bmatrix}0 & 0 & n_4 r^4 + n_6 r^6 + O(r^8)\end{bmatrix}^T$, with $n_4 = \frac{\pi H}{4}$ and $n_6 = \frac{\pi \Delta H}{96}$. The coefficient n_4 agrees with prior work on integral invariants [PWY*07, Theorem 6].

Normal vectors. The integration over $\mathcal{D}_r$ of the normal vector $\mathbf{n}$ of Equation 2 yields

$$
\iint_{\mathcal{D}_r} \mathbf{n}(\rho,\theta)\rho d\rho d\theta = \begin{bmatrix} p_{x4}r^4 + p_{x6}r^6 + O(r^7)\\ p_{y4}r^4 + p_{y6}r^6 + O(r^7)\\ p_{z2}r^2 + p_{z4}r^4 + p_{z6}r^6 + O(r^7)\end{bmatrix}
$$

with the following coefficients

$$
\begin{aligned}
p_{x4} &= -\frac{\pi}{8}(a_{30} + a_{12}),\\
p_{x6} &= \frac{\pi}{48}(a_{30}(2H^2 - K + 4\kappa_1^2) + a_{12}(6H^2 - K) - (a_{50} + 2a_{32} + a_{14})/4),
\end{aligned}
$$

$$
\begin{aligned}
p_{y4} &= -\frac{\pi}{8}(a_{03} + a_{21}),\\
p_{y6} &= \frac{\pi}{48}(a_{03}(2H^2 - K + 4\kappa_2^2) + a_{21}(6H^2 - K) - (a_{41} + 2a_{23} + a_{05})/4),
\end{aligned}
$$

$$
\begin{aligned}
p_{z2} &= \pi,\\
p_{z4} &= -\frac{\pi}{8}(\kappa_1^2 + \kappa_2^2),\\
p_{z6} &= \frac{\pi}{192}(144H^2(H^2 - K) + 24K^2 - 4(a_{22} + a_{40})\kappa_1 - 4(a_{22} + a_{04})\kappa_2 - 3(a_{30}^2 + a_{03}^2) - 2(a_{12}a_{30} + a_{03}a_{21}) - 7(a_{21}^2 + a_{12}^2)).
\end{aligned}
$$

Dot products. The last quantities to integrate are the two dot products introduced in the previous section. Their integrals are $\iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\,.\,\mathbf{n}(\rho,\theta)\rho d\rho d\theta = q_4 r^4 + q_6 r^6 + O(r^8)$ and $\iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\,.\,\mathbf{f}(\rho,\theta)\rho d\rho d\theta = r_4 r^4 + r_6 r^6 + r_8 r^8 + O(r^10)$, with the coefficients equal to

$$
\begin{aligned}
q_4 &= -\frac{\pi H}{4},\\
q_6 &= \frac{\pi}{96}(24H^3 - 16KH - \Delta H)
\end{aligned}
$$

$$
\begin{aligned}
r_4 &= \frac{\pi}{2},\\
r_6 &= \frac{\pi}{24}(3H^2 - K),\\
r_8 &= \frac{\pi}{4068}\left(3(5a_{40} + 6a_{22} + a_{04})\kappa_1 + 3(a_{40} + 6a_{22} + 5a_{04})\kappa_2 + 2\left(5a_{30}^2 + 9a_{21}^2 + 6a_{30}a_{12} + 6a_{03}a_{21} + 9a_{12}^2 + 5a_{03}^2\right)\right).
\end{aligned}
$$

1.3. Algebraic sphere regression

We gather the previous integrals following the smooth version of Equations 2-4 to obtain the asymptotic equivalents of u_c, $\mathbf{u}_\ell$ and u_q of the fitted algebraic sphere. We also give the asymptotic expression of the Pratt's norm (see below Equation 6) in order to obtain the normalized sphere parameters presented in Theorem 1.

Dot products. Before calculating the parameters of the sphere, we need to develop two intermediate expressions. The dot product between the coordinates and the normal vectors integrals is $\iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\rho d\rho d\theta \,.\, \iint_{\mathcal{D}_r} \mathbf{n}(\rho,\theta)\rho d\rho d\theta = s_6 r^6 + s_8 r^8 + O(r^10)$ with $s_6 = \frac{\pi^2 H}{4}$ and $s_8 = \frac{\pi^2}{96}(-12H^3 + 6KH + \Delta H)$.

The second dot product that is applied to the coordinates integral with itself is $\iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\rho d\rho d\theta \,.\, \iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\rho d\rho d\theta = u_8 r^8 + u_{10} r^{10} + O(r^{12})$ with $u_8 = \frac{\pi^2 H^2}{16}$ and $u_{10} = \frac{\pi H \Delta H}{192}$.

Quadratic parameter. To calculate u_q using Equation 4, we rewrite it as a fraction $u_q := \frac{1}{2}\frac{\text{nume}}{\text{deno}}$ with nume the numerator and deno the denominator of u_q (up to the constant $1/2$). In the continuous setting, the numerator of u_q is expressed by

$$\text{nume} := A_r \iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\,.\,\mathbf{n}(\rho,\theta)\rho d\rho d\theta \;-\; \iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\rho d\rho d\theta \,.\, \iint_{\mathcal{D}_r} \mathbf{n}(\rho,\theta)\rho d\rho d\theta, \tag{3}$$

where $A_r = \pi r^2$ is the area of $\mathcal{D}_r$. Its asymptotic polynomials is

$$\text{nume} = v_6 r^6 + v_8 r^8 + O(r^{10}) \tag{4}$$

with $v_6 = -\frac{\pi^2 H}{2}$ and $v_8 = \frac{\pi^2}{48}(18H^3 - 11KH - 2\Delta H)$.

The denominator of u_q is defined as

$$\text{deno} := A_r \iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\,.\,\mathbf{f}(\rho,\theta)\rho d\rho d\theta \;-\; \iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\rho d\rho d\theta \,.\, \iint_{\mathcal{D}_r} \mathbf{f}(r,\theta)\rho d\rho d\theta, \tag{5}$$

which asymptotically leads to

$$\text{deno} = \frac{\pi^2}{2} r^6 \left(1 + w_2 r^2 + w_4 r^4 + O(r^6)\right), \tag{6}$$

with the coefficients

$$w_2 = \frac{1}{24}(3H^2 - 2K),$$
$$w_4 = 3(5a_{40} + 6a_{22} + a_{04})\kappa_1 + 3(a_{40} + 6a_{22} + 5a_{04})\kappa_2 + 2(5a_{30}^2 + 9a_{21}^2 + 6a_{30}a_{12} + 6a_{21}a_{03} + 9a_{12}^2 + 5a_{03}^2).$$

The inverse of the denominator, obtained using the Taylor expansion of $1/(1+X)$, is

$$\frac{1}{\text{deno}} = \frac{2}{\pi^2 r^6}(1 - w_2 r^2 + O(r^4)). \tag{7}$$

Finally, the quadratic parameter u_q of the algebraic sphere is obtained by multiplying nume of Equation 4 and $1/\text{deno}$ of Equation 7. It is asymptotically expressed as

$$u_q = u_{q0} + u_{q2} r^2 + O(r^4) \tag{8}$$

with the following coefficients

$$u_{q0} = -\frac{H}{2},$$
$$u_{q2} = \frac{1}{48}(21H^3 - 13HK - 2\Delta H).$$

Linear parameter. The linear parameter $\mathbf{u}_\ell$ of the sphere is defined as

$$\mathbf{u}_\ell := \frac{1}{A_r}\left(\iint_{\mathcal{D}_r} \mathbf{n}(\rho,\theta) - 2u_q \iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\right) \tag{9}$$

Using previous results of Section 1.2, we obtain

$$\mathbf{u}_\ell = \begin{bmatrix} \mathbf{u}_{\ell x2} r^2 + \mathbf{u}_{\ell x4} r^4 + O(r^5) \\ \mathbf{u}_{\ell y2} r^2 + \mathbf{u}_{\ell y4} r^4 + O(r^5) \\ 1 + \mathbf{u}_{\ell z2} r^2 + \mathbf{u}_{\ell z4} r^4 + O(r^6) \end{bmatrix}, \tag{10}$$

with

$$\mathbf{u}_{\ell x2} = -\frac{a_{30}+a_{12}}{8},$$
$$\mathbf{u}_{\ell y_2} = -\frac{a_{03}+a_{21}}{8},$$
$$\mathbf{u}_{\ell x4} = \frac{1}{48}\left(2(a_{30}+3a_{12})H^2 - (a_{12}+a_{30})K + 4a_{30}\kappa_1^2 - (a_{50}+2a_{32}+a_{14})/4\right),$$
$$\mathbf{u}_{\ell y_4} = \frac{1}{48}\left(2(a_{03}+3a_{21})H^2 - (a_{03}+a_{21})K + 4a_{03}\kappa_2^2 - (a_{41}+2a_{23}+a_{05})/4\right),$$
$$\mathbf{u}_{\ell z2} = -\frac{H^2-K}{4},$$
$$\mathbf{u}_{\ell z4} = \frac{1}{192}\left(165H^4 - 157KH^2 + 24K^2 - 4(a_{40}+a_{22})\kappa_1 - 4(a_{04}+a_{22})\kappa_2 - 3a_{30}^2 - 2a_{12}a_{30} - 7a_{21}^2 - 2a_{03}a_{21} - 7a_{12}^2 - 3a_{03}^2\right),$$

Constant parameter. The constant parameter u_c is defined in the smooth setting by

$$u_c := -\frac{1}{A_r}\left(\mathbf{u}_\ell \,.\iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\rho d\rho d\theta + u_q \iint_{\mathcal{D}_r} \mathbf{f}(\rho,\theta)\,.\,\mathbf{f}(\rho,\theta)\rho d\rho d\theta\right). \tag{11}$$

Its asymptotic expansion is

$$u_c = u_{c4}r^4 + O(r^5) \tag{12}$$

with $u_{c4} = -\frac{1}{96}(9H^3 - 5KH - \Delta H)$.

Pratt's norm. We first give the asymptotic expression of the squared norm of $\mathbf{u}_\ell$, which is required for the Pratt's norm. Starting from Equation. 10, we obtain

$$\|\mathbf{u}_\ell\|^2 = 1 - \frac{H^2-K}{2}r^2 + O(r^3).$$

By using the Taylor polynomial of $\sqrt{1+X}$, we obtain the asymptotic expression of the Pratt's norm

$$p = 1 - \frac{(\kappa_1-\kappa_2)^2}{16}r^2 + O(r^3), \tag{13}$$

which is discussed in Proposition 1.

Using the Taylor polynomial of $1/(1+X)$, we express the inverse of p as

$$\frac{1}{p} = 1 + \frac{(\kappa_1-\kappa_2)^2}{16}r^2 + O(r^3). \tag{14}$$

Normalized sphere parameters To get the asymptotic expressions of the normalized sphere parameters (Equation 6), we multiply each asymptotic expression of the sphere parameters given in Equations 8, 10 and 12 by the inverse of the Pratt's norm given in Equation 14.

We obtain the normalized parameters of the fitted algebraic sphere presented in Theorem 1

$$\hat{u}_q = -\frac{H}{2} + \frac{15H^3 - 7HK - 2\Delta H}{48}r^2 + O(r^3), \tag{15}$$
$$\hat{\mathbf{u}}_\ell = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}\frac{a_{30}+a_{12}}{8}\\ \frac{a_{03}+a_{21}}{8}\\ 0\end{bmatrix} r^2 + O(r^3), \tag{16}$$
$$\hat{u}_c = -\frac{1}{96}(9H^3 - 5KH - \Delta H)r^4 + O(r^5). \tag{17}$$

2. Proof of Theorem 2 - Stability of the mean curvature estimator $\tilde{H}$

This section details the proof of Theorem 2 concerning the stability analysis of the mean curvature estimator $\tilde{H}$ obtained from the algebraic sphere regression.

2.1. Preliminaries on the Gaussian noise

In the asymptotic settings introduced in Section 4.1, the additive noise of Equation 23 amounts to

$$\mathbf{f}^{\star}(x,y) = \mathbf{f}(x,y) + \boldsymbol{\varepsilon}(x,y) \tag{18}$$

where $\mathbf{f}$ is the 'true' surface coordinates given by Equation 9, and $\mathbf{f}^{\star}$ is the 'noisy' surface coordinates. The coordinates of the noise displacement vector $\boldsymbol{\varepsilon}$ follows a Gaussian distribution with zero mean and a standard deviation σ defined by Equation 24. For this theorem, we assume that $\sigma > 2$.

Before detailing the proof, we give some properties required latter on the integrals over $\mathcal{D}_r$ (Equation 11) of several quantities related to the noise model we use

$$\iint_{\mathcal{D}_r} \boldsymbol{\varepsilon} = \mathbf{0}, \tag{19}$$

$$\iint_{\mathcal{D}_r} \|\boldsymbol{\varepsilon}\|^2 = 3\pi\sigma^2 r^2 = 3\pi\delta^2 r^{2\beta+2} + O(r^{2\beta+3}), \tag{20}$$

$$\iint_{\mathcal{D}_r} \|\boldsymbol{\varepsilon}\| = 2\sqrt{2\pi}\sigma r^2 = 2\sqrt{2\pi}\delta r^{\beta+2} + O(r^{\beta+3}). \tag{21}$$

Equations 19, 20, and 21 are obtained from the expected value of a normal, a chi-square, and a chi distribution respectively.

2.2. Stability analysis

The goal of the stability analysis is to inject $\mathbf{f}^{\star}$ (Equation 18) instead of $\mathbf{f}$ in all the equations leading to the mean curvature estimator $\tilde{H}$ (Equation 7).

Quadratic parameter. We analyse the stability of the quadratic parameter u_q of the algebraic sphere using the same ratio formulation $u_q := \frac{1}{2}\frac{\text{nume}}{\text{deno}}$ as in Section 1.

For the numerator, we inject $\mathbf{f}^{\star}$ in Equation 3, which gives

$$\text{nume}^{\star} = \text{nume} + \pi r^2 \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \mathbf{n} - \iint \boldsymbol{\varepsilon}^T \int \boldsymbol{n}.$$

Using Equations 4 and 19, we obtain $\text{nume}^{\star} = -\frac{\pi^2 H}{2} r^6 + O(r^7) + \pi r^2 \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \mathbf{n}$. We develop the remaining integral $\iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \mathbf{n} = \iint_{\mathcal{D}_r} \cos(\theta)\|\boldsymbol{\varepsilon}\|$, where θ is the angle between $\boldsymbol{\varepsilon}$ and $\mathbf{n}$. By bounding $\cos(\theta)$ in $(-1,1)$, and using Equation 21, we obtain $-\frac{\pi^2 H}{2} r^6 + O(r^7) - 2\pi\sqrt{2\pi}\delta r^{\beta+4} + O(r^{\beta+5}) \leq \text{nume}^{\star} \leq -\frac{\pi^2 H}{2} r^6 + O(r^7) + 2\pi\sqrt{2\pi}\delta r^{\beta+4} + O(r^{\beta+5})$. Since $\beta > 2$, then

$$\text{nume}^{\star} = -\frac{\pi^2 H}{2} r^6 + O(r^7),$$

which corresponds to the 6$^{\text{th}}$-order Taylor expansion of nume in Equation 4.

Similarly, we inject $\mathbf{f}^{\star}$ in Equation 5, which gives

$$\text{deno}^{\star} = \text{deno} + \pi r^2 \iint_{\mathcal{D}_r} \|\boldsymbol{\varepsilon}\|^2 + 2\pi r^2 \iint \boldsymbol{\varepsilon}^T \mathbf{f} - \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \left(\iint_{\mathcal{D}_r} \boldsymbol{\varepsilon} + 2 \iint_{\mathcal{D}_r} \mathbf{f} \right).$$

Using Equations 6, 19, and 20, we obtain

$$\text{deno}^{\star} = \frac{\pi^2}{2} r^6 + O(r^7) + 3\pi^2\delta^2 r^{2\beta+4} + O(r^{2\beta+5}) + 2\pi r^2 \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \mathbf{f}.$$

We also develop the remaining integral $\iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \mathbf{f} = \iint_{\mathcal{D}_r} \cos(\varphi)\|\boldsymbol{\varepsilon}\|\|\mathbf{f}\|$, where φ is the angle between $\boldsymbol{\varepsilon}$ and $\mathbf{f}$. By bounding $\cos(\theta)$ in $(-1,1)$ and $\|\mathbf{f}\|$ by $r + O(r^2)$, and using Equation 21, we bound the remaining integral by

$$-2\sqrt{2\pi}\delta r^{\beta+3} + O(r^{\beta+4}) \leq \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \mathbf{f} \leq +2\sqrt{2\pi}\delta r^{\beta+3} + O(r^{\beta+4}) \tag{22}$$

Using these bounds, we obtain $\frac{\pi^2}{2} r^6 + O(r^7) + 3\pi^2\delta^2 r^{2\beta+4} + O(r^{2\beta+5}) - 4\pi\sqrt{2\pi}\delta r^{\beta+5} + O(r^{\beta+6}) \leq \text{deno}^{\star} \leq \frac{\pi^2}{2} r^6 + O(r^7) + 3\pi^2\delta^2 r^{2\beta+4} + O(r^{2\beta+5}) + 4\pi\sqrt{2\pi}\delta r^{\beta+5} + O(r^{\beta+6})$. Since $\beta > 2$, then

$$\text{deno}^{\star} = \frac{\pi^2}{2} r^6 + O(r^7),$$

which corresponds to the 6$^{\text{th}}$-order Taylor expansion of deno in Equation 6.

Finally, since $\beta > 2$ implies $\text{nume}^\star = \text{nume}$ and $\text{deno}^\star = \text{deno}$, then

$$u_q{}^\star = \frac{H}{2} + O(r),$$

which corresponds to u_q (Equation 8).

Linear parameter. The linear parameter $\mathbf{u}_\ell$ defined by Equation 9 is transformed in $\mathbf{u}_\ell{}^\star = \mathbf{u}_\ell - \frac{2u_q{}^\star}{\pi r^2} \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}$. Using Equation 19, we directly obtain $\mathbf{u}_\ell{}^\star = \mathbf{u}_\ell$.

Constant parameter. Injecting $\mathbf{f}^\star$ in u_c (Equation 11) gives

$$u_c{}^\star = u_c - \frac{1}{\pi r^2}\left(\mathbf{u}_\ell{}^{\star T} \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon} + u_q{}^\star \iint_{\mathcal{D}_r} \|\boldsymbol{\varepsilon}\|^2 + 2u_q{}^\star \iint_{\mathcal{D}_r} \boldsymbol{\varepsilon}^T \mathbf{f}\right).$$

Using Equations 12, 19, 20, and 22, we bound $u_c{}^\star$ by $O(r^3) - \sqrt{2\pi}\delta^3 H^2 r^{3\beta+3} + O(r^{3\beta+4}) \leq u_c{}^\star \leq O(r^3) + \sqrt{2\pi}\delta^3 H^2 r^{3\beta+3} + O(r^{3\beta+4})$. Since $\beta > 2$, then $u_c{}^\star = O(r^3)$, which corresponds to the 2^{nd}-order Taylor expansion of u_c (Equation 12).

Mean curvature estimator. We gather the previous results to obtain the perturbed version $\tilde{H}^\star$ of the mean curvature estimator $\tilde{H} := \frac{2u_q}{\sqrt{\|\mathbf{u}_\ell\|^2 - 4u_c u_q}}$ (Equation 7). Since $u_c{}^\star$, $\mathbf{u}_\ell{}^\star$, and $u_q{}^\star$ are similar to their theoretical counterparts u_c, $\mathbf{u}_\ell$, and u_q, these calculations are standard and are thus skipped. We finally obtain $\tilde{H}^\star = \tilde{H}$, which ends the proof of Theorem 2.

3. Proof of Propositions 1, 2 and 3

The proof of Proposition 1 (Pratt's norm) has already been addressed in Section 1, Equation (13).

The asymptotic expansion of the GLS geometric variation given in Proposition 2 is obtained by deriving Equations (12)-(14) with respect to the neighborhood size r, and by combining the results in Equation (18).

Proposition 3 is obtained by simply injecting Equations (12)-(14) inside Equation (21) to express the projection operator presented in Equation (22).

4. Additional results

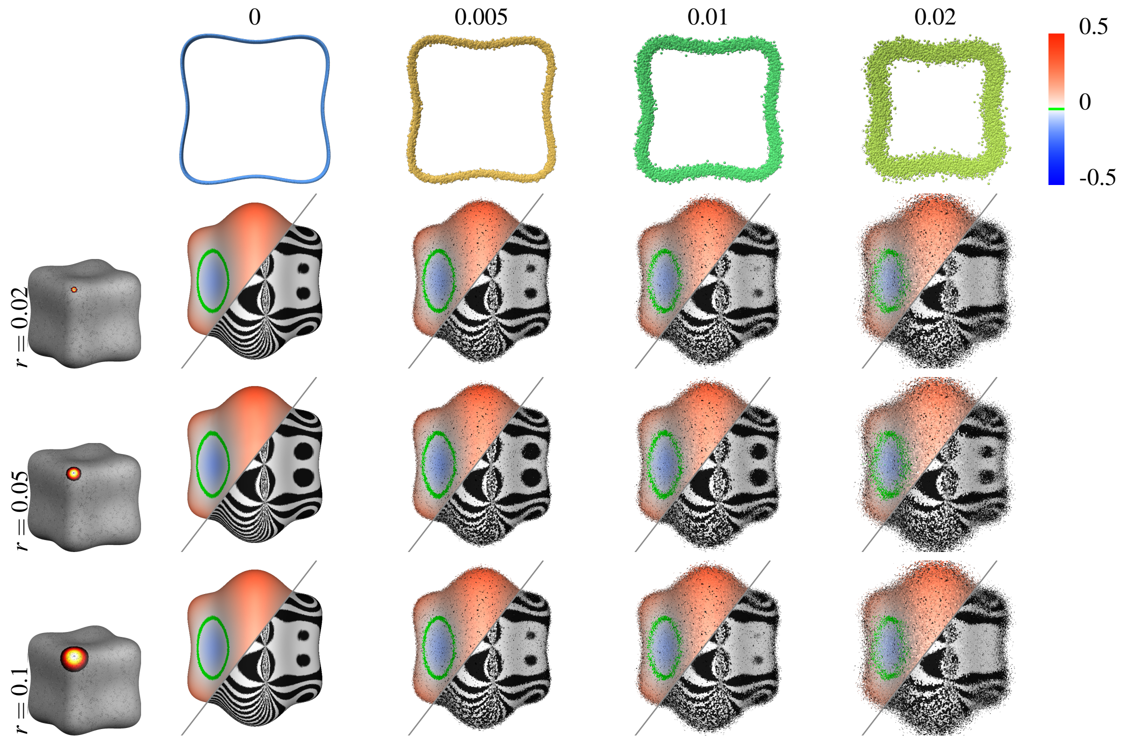

Figure 1: *Additional results for* `APSS` *for various radii r (with signed mean curvature estimation).*

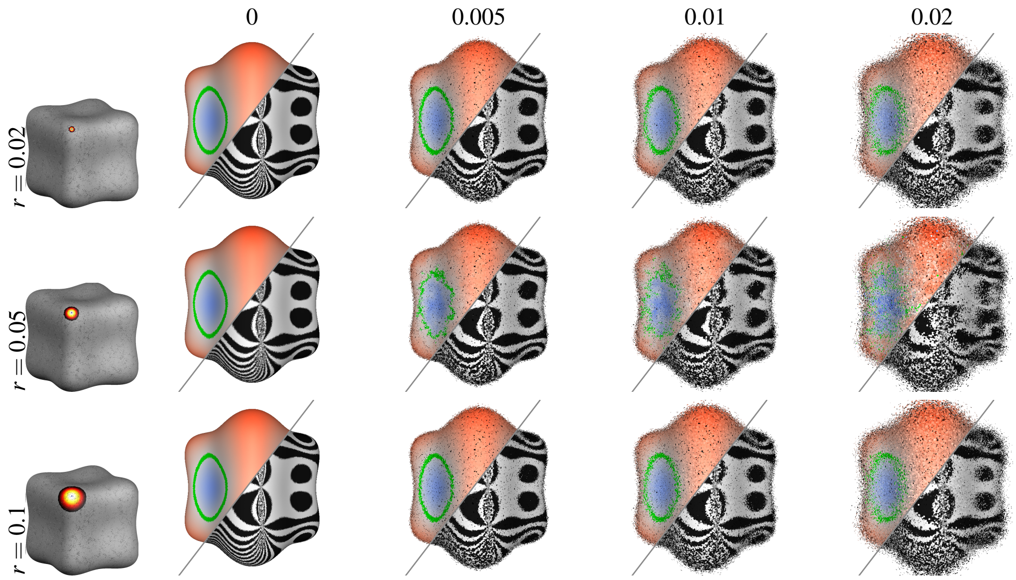

Figure 2: *Additional results for* `ASO` *for various radii r (with signed mean curvature estimation).*

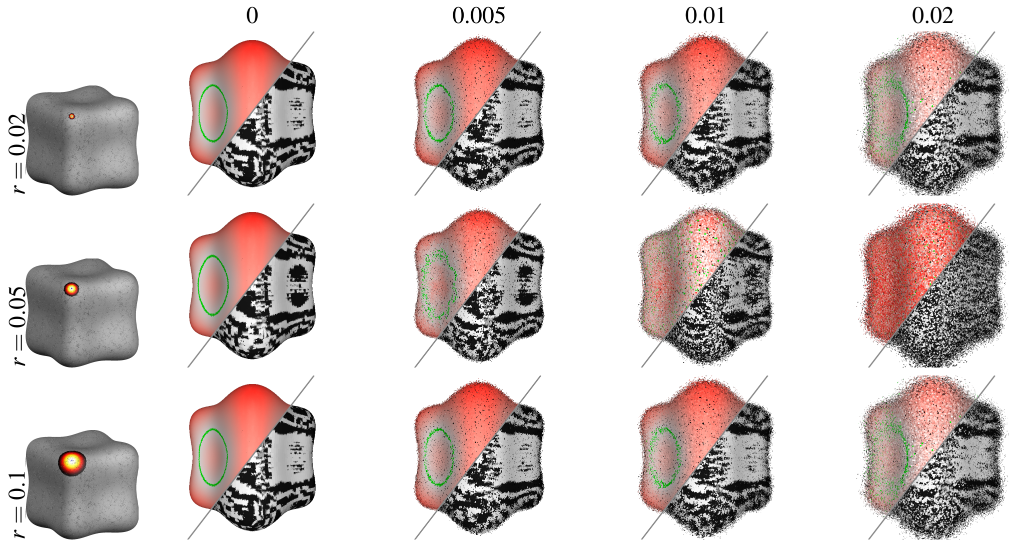

Figure 3: *Additional results for* `OJets` *for various radii r (absolute value of mean curvature).*

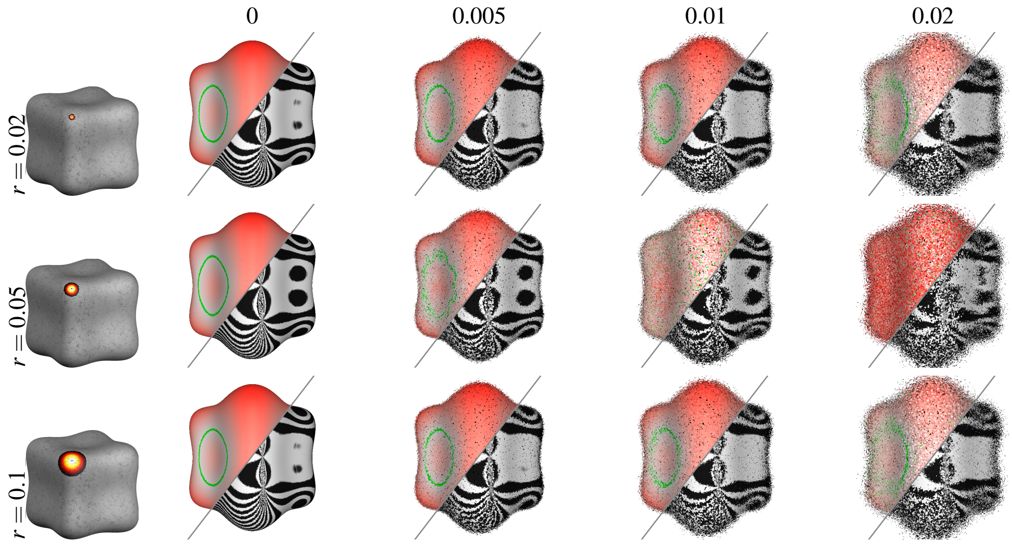

Figure 4: *Additional results for* `WJets` *for various radii r (absolute value of mean curvature).*

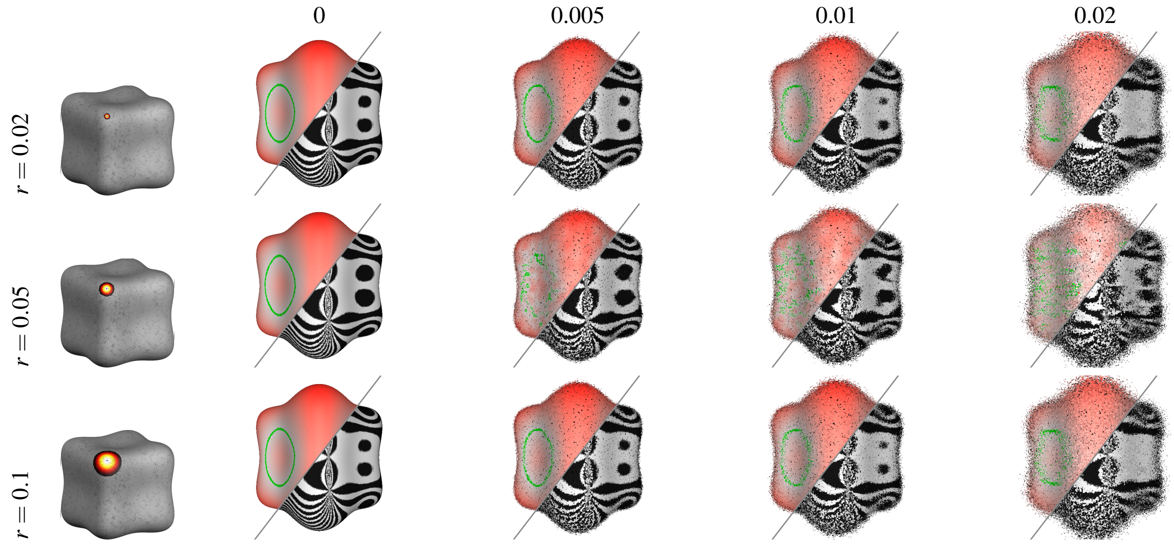

Figure 5: *Additional results for* `PSS` *for various radii r (absolute value of mean curvature).*

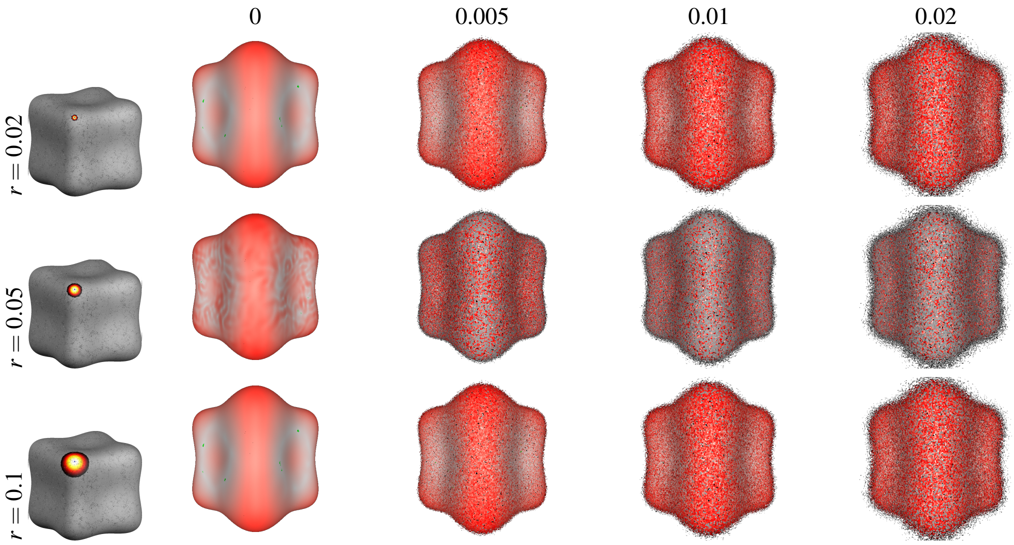

Figure 6: *Additional results for the distance-to-barycenter mean curvature estimation from [PWY*07] for various radii r (absolute value of mean curvature).*

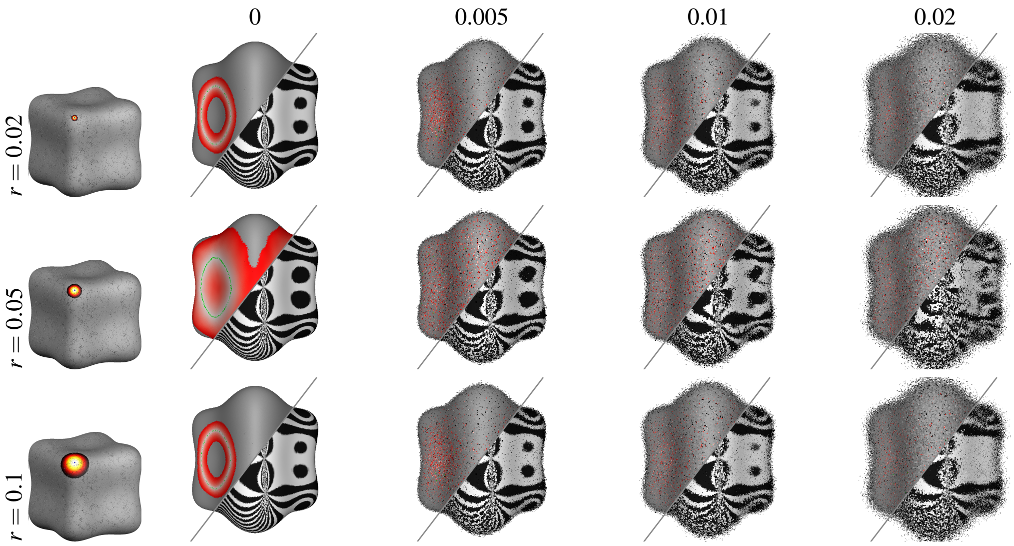

Figure 7: *Additional results for the distance-to-plane mean curvature estimation from [DMSL11] for various radii r (absolute value of mean curvature).*

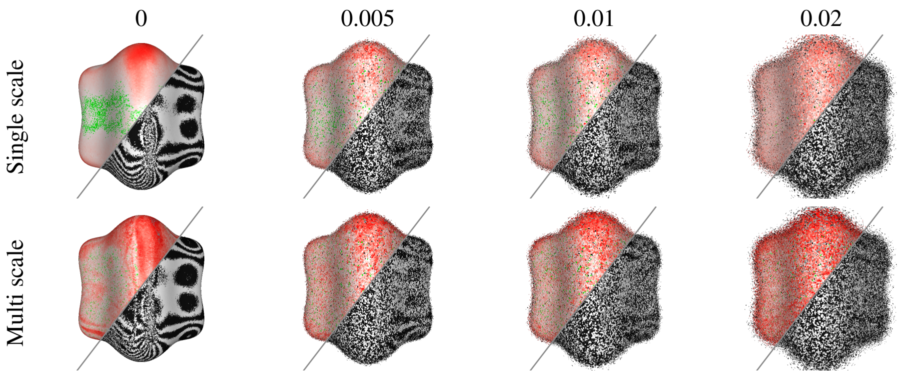

Figure 8: *Additional results for PCPNET [GKOM18] using single and multi scale pretrained networks as provided by the authors.*

Figure 9: *Mean curvature and normal vector estimations on an highly non-uniform sampling of the Goursat's surface.*

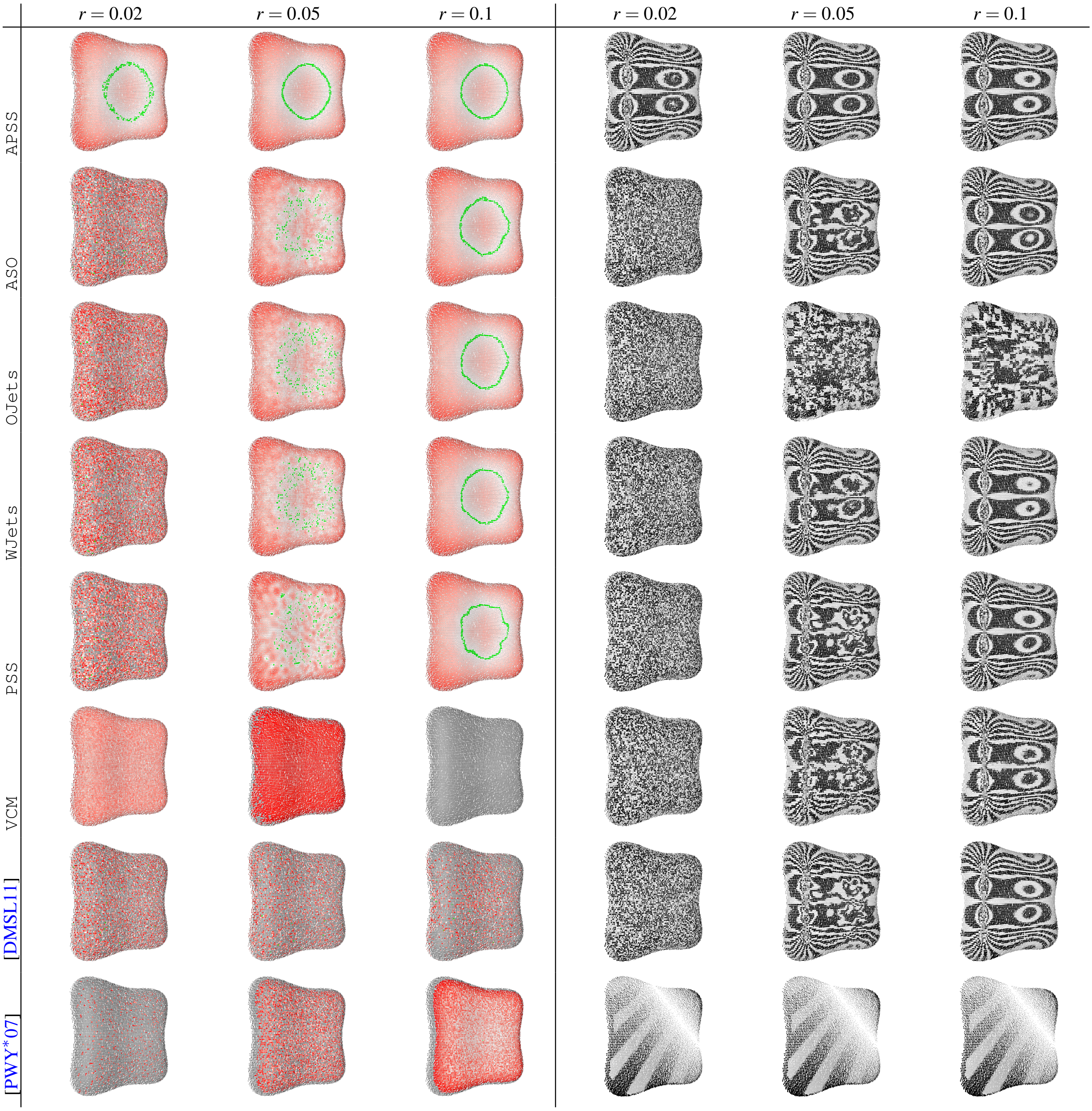

Figure 10: *Mean curvature and normal vector estimations on a non-uniform Lidar-like sampling strategy: from a source points, we regularly sample the sphere of directions and shoot rays that intersect the surface (with an additional Gaussian noise with* $\sigma = 0.013$ *at the intersection point along that ray), 14850 samples.*